First-Order Logic

Part Two

Recap from Last Time

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about many objects at once.

Some muggle is intelligent.

$$\exists m.\ (Muggle(m) \land Intelligent(m))$$

∃ is the ***existential quantifier*** and says "for some choice of m, the following is true."

"For any natural number n,
n is even if and only if n^2 is even"

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the ***universal quantifier*** and says "for any choice of n, the following is true."

“Some *P* is a *Q*”

translates as

$$\exists x.\ (P(x) \land Q(x))$$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x. (P(x) \land Q(x))$$

If x is an example, it *must* have property P on top of property Q.

“All *P*'s are *Q*'s”

translates as

$$\forall x.\ (P(x) \to Q(x))$$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x.\ (P(x) \to Q(x))$$

If x is a counterexample, it *must* have property P but not have property Q.

New Stuff!

The Aristotelian Forms

"All *A*s are *B*s"

$$\forall x.\ (A(x) \to B(x))$$

"Some *A*s are *B*s"

$$\exists x.\ (A(x) \land B(x))$$

"No *A*s are *B*s"

$$\forall x.\ (A(x) \to \neg B(x))$$

"Some *A*s aren't *B*s"

$$\exists x.\ (A(x) \land \neg B(x))$$

It is worth committing these patterns to memory. We'll be using them throughout the day and they form the backbone of many first-order logic translations.

The Art of Translation

Using the predicates

- *Person*(*p*), which states that *p* is a person, and
- *Loves*(*x*, *y*), which states that *x* loves *y*,

write a sentence in first-order logic that means "every person loves someone else."

Answer at

https://cs103.stanford.edu/pollev

Every person loves someone else

Every person loves some other person

Every person p loves some other person

Every person p loves some other person

“All As are Bs”

$$\forall x.\ (A(x) \to B(x))$$

$\forall p.\ (Person(p) \rightarrow$

p loves some other person

$)$

"All *A*s are *B*s"

$\boldsymbol{\forall x.\ (A(x) \rightarrow B(x))}$

$\forall p. \ (Person(p) \rightarrow$

p loves some other person

$)$

$\forall p. \ (Person(p) \to$

there is some other person that p loves

$)$

$\forall p. \, (Person(p) \rightarrow$

there is a person other than p that p loves

$)$

$\forall p. \, (Person(p) \rightarrow$

there is a person q, other than p, where p loves q

$)$

$\forall p. \; (Person(p) \rightarrow$

there is a person q, other than p, where

p loves q

$)$

$\forall p. \, (Person(p) \rightarrow$

there is a person q, other than p, where

p loves q

$)$

"Some *A*s are *B*s"

$\exists x. \, (A(x) \land B(x))$

$\forall p.\ (Person(p) \rightarrow$

$\quad \exists q.\ (Person(q)\ \wedge$*, other than p, where*

$\quad\quad$ *p loves q*

$\quad)$

$)$

"Some *A*s are *B*s"

$\boldsymbol{\exists x.\ (A(x)\ \wedge\ B(x))}$

$\forall p.\ (Person(p) \rightarrow$

$\quad \exists q.\ (Person(q)\ \land$ *, other than p, where*

p loves q

$\quad)$

$)$

$\forall p.\ (Person(p) \to$

$\quad \exists q.\ (Person(q) \land p \neq q \land$

$\qquad$ *p loves q*

$\quad)$

$)$

$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

Using the predicates

- *Person*(*p*), which states that *p* is a person, and
- *Loves*(*x*, *y*), which states that *x* loves *y*,

write a sentence in first-order logic that means “there is a person that everyone else loves.”

Answer at

[***https://cs103.stanford.edu/pollev***](https://cs103.stanford.edu/pollev)

There is a person that everyone else loves

There is a person p where everyone else loves p

There is a person p where everyone else loves p

"Some *A*s are *B*s"

$\exists x.\ (A(x) \land B(x))$

$\exists p.\ (Person(p)\ \land$

everyone else loves p

$)$

> “Some *A*s are *B*s”
>
> $\boldsymbol{\exists x.\ (A(x)\ \land\ B(x))}$

$\exists p. \ (Person(p) \ \wedge$

everyone else loves p

$)$

$\exists p. \ (Person(p) \ \wedge$

every other person q loves p

$)$

$\exists p.\ (Person(p)\ \wedge$

every person q, other than p, loves p

$)$

$\exists p. \ (Person(p) \ \land$

every person q, other than p, loves p

$)$

"All *A*s are *B*s"

$\forall x. \ (A(x) \rightarrow B(x))$

$\exists p. \, (Person(p) \land$

$\quad \forall q. \, (Person(q) \land p \neq q \rightarrow$

$\qquad$ *q loves p*

$\quad)$

$)$

"All *A*s are *B*s"

$$\boldsymbol{\forall x. \, (A(x) \rightarrow B(x))}$$

$\exists p.\ (Person(p)\ \land$

$\quad \forall q.\ (Person(q)\ \land\ p \neq q\ \rightarrow$

q loves p

$\quad)$

$)$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow \\
&\qquad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: "Every person loves someone else"

For every person… $\forall p.\ (Person(p) \to$

… there is another person … $\exists q.\ (Person(q) \land p \neq q \land$

… they love $Loves(p, q)$

$)$

$)$

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: "There is someone everyone else loves."

There is a person…

… that everyone else …

… loves.

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow \\
&\quad\quad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

For Comparison

For every person…

… there is another person …

… they love

$$\forall p.\ (Person(p) \to \\ \quad \exists q.\ (Person(q) \land p \neq q \land \\ \quad\quad Loves(p, q) \\ \quad) \\ )$$

There is a person…

… that everyone else …

… loves.

$$\exists p.\ (Person(p) \land \\ \quad \forall q.\ (Person(q) \land p \neq q \to \\ \quad\quad Loves(q, p) \\ \quad) \\ )$$

Restricted Quantifiers

Quantifying Over Sets

- The notation

$$\forall x \in S.\ P(x)$$

means "for any element x of set S, $P(x)$ holds." (It's vacuously true if S is empty.)

- The notation

$$\exists x \in S.\ P(x)$$

means "there is an element x of set S where $P(x)$ holds." (It's false if S is empty.)

Quantifying Over Sets

- The syntax

$$\forall x \in S.\ P(x)$$

$$\exists x \in S.\ P(x)$$

  is allowed for quantifying over sets.

- In CS103, feel free to use these restricted quantifiers, but please do not use variants of this syntax.
- For example, don't do things like this:

⚠ $\forall x \text{ with } P(x).\ Q(x)$ ⚠

⚠ $\forall y \text{ such that } P(y) \land Q(y).\ R(y).$ ⚠

⚠ $\exists P(x).\ Q(x)$ ⚠

Quantifier Ordering

For Comparison

For every person… $\forall p.\ (Person(p) \to$

… there is another person … $\exists q.\ (Person(q) \land p \neq q \land$

… they love $Loves(p, q)$

$)$

$)$

There is a person… $\exists p.\ (Person(p) \land$

… that everyone else … $\forall q.\ (Person(q) \land p \neq q \to$

… loves. $Loves(q, p)$

$)$

$)$

Quantifier Ordering

- Consider these two first-order formulas:

$$\forall m \in \mathbb{N}.\ \exists n \in \mathbb{N}.\ m < n.$$

$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ m < n.$$

- One of these statements is true. One is false. Which is which?
- Why?

Quantifier Ordering

- Consider these two first-order formulas:

$$\forall m \in \mathbb{N}.\ \exists n \in \mathbb{N}.\ m < n.$$

$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ m < n.$$

- This says

for every natural number *m*,
there's a larger natural number *n*.

- This is true: given any $m \in \mathbb{N}$, we can choose n to be $m + 1$.
- Notice that we can pick n based on m, and we don't have to pick the same n each time.

Quantifier Ordering

- Consider these two first-order formulas:

$$\forall m \in \mathbb{N}.\ \exists n \in \mathbb{N}.\ m < n.$$

$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ m < n.$$

- This says

there is a natural number *n*
that's larger than every *m* ∈ ℕ

- This is false: no natural number is bigger than every natural number.
- Because $\exists n \in \mathbb{N}$ comes first, we have to make a single choice of n that works.

Quantifier Ordering

- The statement

$$\forall x.\ \exists y.\ P(x, y)$$

  means “for any choice of *x*, there's some choice of *y* where *P*(*x*, *y*) is true.”
- The choice of *y* can be different every time and can depend on *x*.

Quantifier Ordering

- The statement

$$\exists x.\ \forall y.\ P(x, y)$$

  means "there is some x where for any choice of y, we get that $P(x, y)$ is true."
- Since the inner part has to work for any choice of y, this places a lot of constraints on what x can be.

Order matters when mixing existential and universal quantifiers!

Time-Out for Announcements!

Problem Set Two

- Problem Set One was due today at 1:00PM.
  - You can extend the deadline to 1:00PM Saturday using one of your late days. As usual, no late submissions will be accepted beyond 1:00PM Saturday without prior approval.
- Problem Set Two goes out today. It's due next Friday at 1:00PM.
  - Explore first-order logic!
  - Expand your proofwriting toolkit!
- We have some online readings for this problem set.
  - Check out the ***Guide to Logic Translations*** for more on how to convert from English to FOL.
  - Check out the ***Guide to Negations*** for information about how to negate formulas.
  - Check out the ***First-Order Translation Checklist*** for details on how to check your work.

Back to CS103!

Mechanics: Negating Statements

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	There is an x where $P(x)$ is false.
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	For all objects x, $P(x)$ is false.
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	There is an x where $P(x)$ is true.
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects *x*, *P*(*x*) is true.	**There is an *x* where *P*(*x*) is false.**
$\exists x.\ P(x)$	There is an *x* where *P*(*x*) is true.	For all objects *x*, *P*(*x*) is false.
$\forall x.\ \neg P(x)$	For all objects *x*, *P*(*x*) is false.	There is an *x* where *P*(*x*) is true.
$\exists x.\ \neg P(x)$	**There is an *x* where *P*(*x*) is false.**	For all objects *x*, *P*(*x*) is true.

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	For all objects x, $P(x)$ is false.
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	There is an x where $P(x)$ is true.
$\exists x.\ \neg P(x)$	**There is an x where $P(x)$ is false.**	For all objects x, $P(x)$ is true.

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
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	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	**For all objects x, $P(x)$ is false.**
$\forall x.\ \neg P(x)$	**For all objects x, $P(x)$ is false.**	There is an x where $P(x)$ is true.
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	**$\forall x.\ \neg P(x)$**
$\forall x.\ \neg P(x)$	**For all objects x, $P(x)$ is false.**	There is an x where $P(x)$ is true.
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\boldsymbol{\exists x.\ \neg P(x)}$
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	$\boldsymbol{\forall x.\ \neg P(x)}$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	There is an x where $P(x)$ is true.
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	**There is an x where $P(x)$ is true.**	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	**There is an x where $P(x)$ is true.**
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	**There is an x where $P(x)$ is true.**	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	**For all objects *x*, *P*(*x*) is true.**	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	There is an *x* where *P*(*x*) is true.	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects *x*, *P*(*x*) is false.	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	There is an *x* where *P*(*x*) is false.	**For all objects *x*, *P*(*x*) is true.**

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	**For all objects *x*, *P*(*x*) is true.**	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	There is an *x* where *P*(*x*) is true.	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects *x*, *P*(*x*) is false.	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	There is an *x* where *P*(*x*) is false.	$\forall x.\ P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	$\forall x.\ P(x)$

Negating First-Order Statements

- Use the equivalences

  $$\neg\forall x.\ A \quad \textit{is equivalent to} \quad \exists x.\ \neg A$$

  $$\neg\exists x.\ A \quad \textit{is equivalent to} \quad \forall x.\ \neg A$$

  to negate quantifiers.
- Mechanically:
  - Push the negation across the quantifier.
  - Change the quantifier from ∀ to ∃ or vice-versa.
- Use techniques from propositional logic to negate connectives.

Taking a Negation

$$\forall x.\ \exists y.\ Loves(x, y)$$

("Everyone loves someone.")

$$\neg\forall x.\ \exists y.\ Loves(x, y)$$
$$\exists x.\ \neg\exists y.\ Loves(x, y)$$
$$\exists x.\ \forall y.\ \neg Loves(x, y)$$

("There's someone who doesn't love anyone.")

Two Useful Equivalences

- The following equivalences are useful when negating statements in first-order logic:

$$\neg(p \land q) \quad \textit{is equivalent to} \quad p \to \neg q$$

$$\neg(p \to q) \quad \textit{is equivalent to} \quad p \land \neg q$$

- These identities are useful when negating statements involving quantifiers.
  - $\land$ is used in existentially-quantified statements.
  - $\to$ is used in universally-quantified statements.
- When pushing negations across quantifiers, we *strongly recommend* using the above equivalences to keep $\to$ with $\forall$ and $\land$ with $\exists$.

Negating Quantifiers

- What is the negation of the following statement, which says "there is a cute puppy"?

$$\boldsymbol{\exists x.\ (Puppy(x) \land Cute(x))}$$

- We can obtain it as follows:

$$\neg\exists x.\ (Puppy(x) \land Cute(x))$$

$$\forall x.\ \neg(Puppy(x) \land Cute(x))$$

$$\forall x.\ (Puppy(x) \to \neg Cute(x))$$

- This says "no puppy is cute."
- Do you see why this is the negation of the original statement from both an intuitive and formal perspective?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

("There is a set with no elements.")

$$\neg\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ \neg(Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \neg\forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ \neg\neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ x \in S)$$

("Every set contains at least one element.")

Expressing Uniqueness

Using the predicate

- *WayToFindOut*(w), which states that w is a way to find out,

write a sentence in first-order logic that means “there is only one way to find out.”

There is only one way to find out.

Something is a way to find out, and nothing else is.

Some thing w is a way to find out, and nothing else is.

Some thing w is a way to find out, and nothing besides w is a way to find out

$\exists w. \, (WayToFindOut(w) \, \wedge$

nothing besides w is way to find out

$)$

$\exists w. \, (WayToFindOut(w) \land$

anything that isn't w isn't a way to find out

$)$

$\exists w. \ (WayToFindOut(w) \ \wedge$

any thing x that isn't w isn't a way to find out

$)$

$\exists w.\ (WayToFindOut(w)\ \land$

$\quad \forall x.\ (x \neq w \rightarrow$ *x isn't a way to find out*$)$

$)$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\quad \forall x.\ (x \neq w \rightarrow \neg \mathit{WayToFindOut}(x)) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w) \land \\
&\quad \forall x.\ (x \neq w \rightarrow \neg \mathit{WayToFindOut}(x)) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\quad \forall x.\ (\mathit{WayToFindOut}(x) \rightarrow x = w) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\quad \forall x.\ (\mathit{WayToFindOut}(x) \rightarrow x = w) \\
&)
\end{aligned}
$$

Expressing Uniqueness

- To express the idea that there is exactly one object with some property, we write that
  - there exists at least one object with that property, and that
  - there are no other objects with that property.
- You sometimes see a special "uniqueness quantifier" used to express this:

$$\exists! x.\ P(x)$$

- For the purposes of CS103, please do not use this quantifier. We want to give you more practice using the regular ∀ and ∃ quantifiers.

Next Time

- ***Functions***
  - How do we model transformations and pairings?
- ***First-Order Definitions***
  - Where does first-order logic come into all of this?
- ***Proofs with Definitions***
  - How does first-order logic interact with proofs?